Ngā Whetū Resources

Calculus Year 13 (Level 8)

Summary 9

Integration

If $f(x) = g(x) + h(x)$ then $\int f(x)dx = \int g(x)dx + \int h(x)dx$. Finding an integral therefore often means to convert a product or quotient into a sum or difference.

Unlike with differentiation, there is usually no direct way to integrate a function. Frequently you make an intelligent guess and then check the answer by differentiation to see if you obtain the original function under the integral (integrand). Then make corrections if necessary.

Indefinite integrals
If no bounds or boundary conditions are given the result of an integration is only known up to an unknown constant *c*

INTEGRATION METHODS

Polynomial form $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c \ (n \neq -1)$

Ex. $\int (3x^2 - x)dx = \frac{3x^3}{3} - \frac{x^2}{2} + c = x^3 - \frac{1}{2}x^2 + c$

Trigonometric functions

- Use table for differentiation in formula sheet in reversed direction
- convert powers and products of trig functions into sums using trig formulas

Ex. $\int \cos^2 x dx = \int (\frac{1}{2}\cos(2x) + \frac{1}{2})dx$ (using $\cos 2A = 2\cos^2 A - 1$)

now integrate this $\frac{1}{2}\sin(2x) + \frac{1}{2}x + c$ Now check this by differentiation. The first term in the result must be corrected by the additional factor $\frac{1}{2}$ so the answer is

$$\int \cos^2 x dx = \int (\frac{1}{2}\cos(2x) + \frac{1}{2})dx = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

Exponential functions
Try same function as first result. Then differentiate and correct with a factor if necessary.

Ex. $\int \frac{e^{5x} - e^{2x}}{e^{3x}} dx = \int (e^{2x} - e^{-x})dx$ Try $e^{2x} - e^{-x} + c$ differentiate and check. In the first term we need a factor of $\frac{1}{2}$ and in the second term a factor of -1.

So the answer is $\frac{1}{2}e^{2x} + e^{-x} + c$

Form $\frac{1}{x}$

Take the (absolute value of) the logarithm of the denominator, differentiate and check.

Ex. $\int \frac{5}{2x+1}dx$. So try $5\ln|2x+1|+c$ Differentiation shows that we need a factor of $\frac{1}{2}$ so the answer is $\frac{5}{2}\ln|2x+1|+c$

Note that the integration constant *c* can also be written as $\ln k$ and be brought under the logarithm hence as $\frac{5}{2}\ln|k(2x+1)|$ (remember $\ln k+\ln x=\ln kx$)

"Inverted Chain Rule"

When a **product** under the integral is in the form $\int f'(x)\times f(x)dx$ then just integrate $f(x)$.

When differentiating, the chain rule will automatically give the factor $f'(x)$. Of course you need to check for necessary factors.

Ex. $\int 4x(x^2-1)^3dx$. Integrate $(x^2-1)^3$ giving $\frac{(x^2-1)^4}{4}+c$.

Differentiation gives $\frac{4}{4}(x^2-1)^3\times 2x$. We only need an extra factor 2 so the answer is

$$2\frac{(x^2-1)^4}{4}+c=\frac{1}{2}(x^2-1)^4+c$$

If you recognise the **quotient** form $\int \frac{f'(x)}{f(x)}dx$ then integrate $\int \frac{1}{f(x)}dx=\ln|x|+c$ and correct with a factor if necessary.

Ex. $\int \frac{3x^2+2}{x^3+2x+1}dx$. Try $\ln|x^3+2x+1|+c$. Differentiation gives $\frac{3x^2+2}{x^3+2x+1}$ and no correction is necessary. So the answer is $\ln|x^3+2x+1|+c$ or $\ln|k(x^3+2x+1)|$

Rational functions of the form $\frac{ax+b}{cx+d}$

Convert this to a sum by long division then integrate as usual.

Ex. $\int \frac{2x+3}{x+2}dx=\int (2-\frac{1}{x+2})dx=2x-\ln|x+2|+c$

Algebraic substitution

This is for products (or quotients) which are not of the form $f'(x)f(x)$ (or $\frac{f'(x)}{f(x)}$).

Substitute a parameter (*t*) for one of the factors, solve for *x* and differentiate to find $\frac{dx}{dt}$. The integrand can then be written as a sum which can be integrated with any of the other techniques.

Ex. $\int x(3x+2)^5\,dx$. Let $t=3x+2$ so $x=\frac{t}{3}-\frac{2}{3}$. Hence $\frac{dx}{dt}=\frac{1}{3}$ and thus $dx=\frac{1}{3}dt$.

Now substitute: $\int x(3x+2)^5\,dx=\int\left(\frac{t}{3}-\frac{2}{3}\right)(t)^5\,dx=\int\left(\frac{t}{3}-\frac{2}{3}\right)(t)^5\,\frac{1}{3}dt=\int\left(\frac{t^6}{9}-\frac{2t^5}{9}\right)dt$.

This can now be integrated to $\frac{t^7}{63}-\frac{t^6}{27}+c$ and with back-substitution $\frac{(3x+2)^7}{63}-\frac{(3x+2)^6}{27}+7$

DEFINITE INTEGRATION

Integration of a function between given limits makes it possible to find a unique answer because the integration constant is eliminated:

If $F'(x)=f(x)$ then $\int_a^b f(x)dx=\big(F(b)+c\big)-\big(F(a)+c\big)=F(b)-F(a)$

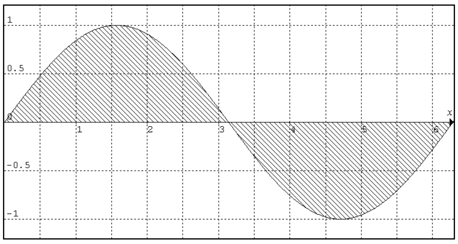
Ex. $\int_1^3 (x^2+2)dx=\frac{1}{3}x^3+2x\Big|_1^3=\left(\frac{1}{3}3^3+2\cdot 3\right)-\left(\frac{1}{3}1^3+2\cdot 1\right)=15-\frac{7}{3}=12\frac{2}{3}$

Practice all these techniques with exercises in the book.

Area

A definite integral determines the area between a function and the x-axis.

Ex. $\int_0^{\pi}\sin(x)dx=-\cos(x)\big|_0^{\pi}=-\cos(\pi)-(-\cos(0))=1+1=2$



But

$\int_0^{2\pi}\sin(x)dx=-\cos(x)\big|_0^{2\pi}=-\cos(2\pi)-(-\cos(0))=-1+1=0$

And

$\int_{\pi}^{2\pi}\sin(x)dx=-\cos(x)\big|_{\pi}^{2\pi}=-\cos(2\pi)-(-\cos(\pi))=-1-1=-2$

Because area determined by integration means signed area. Part(s) below the axis are considered negative. Hence if a function has x-intercepts within the interval of integration, integrate between these intercepts.

Y-axis as boundary: Make x the subject and integrate with respect to y

Ex. $y=\ln(x)\Rightarrow x=e^y$ Solve $\int_a^b e^y dy$

Between function and line: Translate the function so that the line is on the x-axis

Ex. Area between $y=4-x^2$ and the line $y=2$ then integrate shifted function

$\int_a^b (4-x^2-2)dx=\int_a^b (2-x^2)dx$

Between two functions: Take the difference of two definite integrals (areas):

Ex. Area between $y = e^x$ and $y = x$ is $\int_a^b e^x dx - \int_a^b x dx$ but beware for signed area. Always good to draw a sketch of the situation.

Volume of revolution

Formed when an area is rotated 360 degrees about x- or y-axis or about a line paralel to either axis.

General form: $V = \pi \int_a^b f(x)^2 dx$

Ex. Volume between 0 and 9 when $y = x^2$ is rotated about y-axis: $y = x^2 \Rightarrow x = \sqrt{y}$

Integrate $\pi \int_0^9 \left(\sqrt{y}\right)^2 dy = \pi \int_0^9 y dy = \frac{\pi}{2} y^2 \Big|_0^9 = \frac{81}{2}\pi$

NUMERICAL INTEGRATION

Many functions cannot be integrated algebraically (meaning exact). Then you can use an approximation method of which there are many. We are using two methods:

Trapezium rule Choose the number of intervals (*n*) Then interval width is given by $h = \frac{x_n - x_0}{n}$. Now determine the area of successive trapeziums of width *h* and function-value as heights.

Formula: $\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_n) + 2\left\{ f(x_1) + f(x_2) + ... + f(x_{n-1}) \right\} \right]$

Ex. Estimate area under $y = \frac{1}{x}$ between $x = 1$ and $x = 3$ in 4 intervals:

$\int_1^3 \frac{1}{x} dx = \frac{0.5}{2}\left[\frac{1}{1} + \frac{1}{3} + 2\left\{ \frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5} \right\} \right] = 1.117$ (3dp)

Simpson's Rule Same procedure but different formula (only even number of intervals):

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\left\{ f(x_1) + f(x_3) + ... + f(x_{n-1}) \right\} + 2\left\{ f(x_2) + f(x_4) + ... + f(x_{n-2}) \right\} \right]$$

Ex. Estimate area under $y = \sqrt{x}$ between $x = 0$ and $x = 1$ in 4 intervals:

$\int_0^1 \sqrt{x} dx = \frac{0.25}{3}\left[\sqrt{0} + \sqrt{1} + 4\left\{ \sqrt{0.25} + \sqrt{0.75} \right\} + 2\left\{ \sqrt{0.5} \right\} \right] = 0.657$ (3dp)