Ngā Whetū Resources

Calculus Year 13 (Level 8)

Differential Equations

So far we have done mathematics with functions like $y = f(x)$ which could describe any practical situation we want to investigate. However many problems in real life (sciences, engineering, etc.) cannot be expressed in this way. In those cases we can only describe how a process **changes** rather than what the process itself looks like exactly. In mathematics we use **differential equations** (**DE**'s) to describe and solve those situations.

A differential equation involves derivatives. $\frac{dy}{dx} = x + 5$ for example is a differential equation. It is a first-order DE because only the first derivative appears. (There are also second-order DE's such as $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$). Going back to the first equation, it describes a curve with a gradient in any point equal to the x-value plus 5. The only way to solve an equation like this is to integrate. This will produce an unknown constant of integration so we can predict that a DE in general does not have one unique solution. Therefore we need boundary conditions to uniquely solve a DE. This will be explained with a number of typical examples.

GEOMETRIC PROBLEM

This example doesn't yet show you real life applications but we give this first to explain the procedure.

Let's take the example above: $\boxed{\frac{dy}{dx} = x + 5}$

Step 1 **Separate variables:**

You can treat the differential $\frac{dy}{dx}$ as an ordinary fraction as we have done before. Separate *y* and *x* by re-arranging to $dy = (x+5)dx$. All parts containing *y* are on the left and all parts containing *x* are on the right.

Step 2 **Integrate both sides.**

We integrate the left hand side to *y* and the right hand side to *x*, so $\int dy = \int (x+5)dx$ or $y + c_1 = \frac{1}{2}x^2 + 5x + c_2$. We can take both integration constants together and write $y = \frac{1}{2}x^2 + 5x + C$. We have obtained a family of curves all satisfying the original DE. If you differentiate this general solution you obtain the original DE. (Verify this). But we cannot use this solution because we don't know the value of the constant *C*.

Step 3 **Find the constant from a boundary condition.**

An example of a boundary condition could be that we know one point of the function. E.g. the curve passes through the point (-4,-1). Substituting gives $-1=\frac{1}{2}\times 16-5\times 4+C$ or $C=20-8-1=11$. Hence this particular solution to the DE is $\boxed{y=\frac{1}{2}x^2+5x+11}$

Hint: Illustrate this example yourself using Graphmatica. First type the DE in the form : $dy=x+5$ In this way Graphmatica understands this is a DE and it will display the general form of this family of parabola's. Now plot the specific result $\boxed{y=\frac{1}{2}x^2+5x+11}$ and you will see that this parabola follows the general pattern (it is a member of the family) and also passes through the point (-4,-1).

Now let's go into "real life" problems.

A BOAT MOVING THROUGH WATER.

The engine of a boat provides a thrust force of 2000 N. The water causes a resistance force in opposite direction of $100v$ where v is the speed of the boat. (This is close to physical reality: that resistance is a linear function of the speed). The mass of the boat is 1000 kg. A fundamental law of physics states that under the influence of a force *F* an object of mass *m* accelerates as $\frac{dv}{dt}=\frac{F}{m}$ (usually written as $F=ma$). Using the values given we obtain: $\frac{dv}{dt}=\frac{2000-100v}{1000}=2-0.1v$ This is a DE describing how the speed changes with time.

Let us follow the 3 step approach:

Step 1 **Separate variables:**

$$\frac{1}{2-0.1v}dv=dt$$

Step 2 **Integrate both sides.**

$\int\frac{1}{2-0.1v}dv=\int dt$ or $\frac{\ln|2-0.1v|}{-0.1}=t+C$ or $\ln|2-0.1v|=-0.1t+\ln K$ (we converted the constant C into K by defining $\ln K=-0.1C$) Often we obtain a logarithm when we integrate a DE and we can eliminate this by taking the e-function on both sides: $2-0.1v=e^{-0.1t+\ln K}=e^{-0.1t}\times e^{\ln K}=Ke^{-0.1t}$. Make *v* subject: $0.1v=2-Ke^{-0.1t}$ or $v=20-10Ke^{-0.1t}$. K is arbitrary so we can simplify this with a new constant ($A=10K$) to $\boxed{v=20-Ae^{-0.1t}}$.

We used a bit of algebra here but this is the simplest form to describe the speed as a function of time. It is the solution of the original DE but we still have the unknown constant A.

Step 3 **Find the constant from a boundary condition.**

We define the speed $v=0$ at $t=0$. Substitute this in the solution:

$0=20-Ae^0=20-A$ Hence $A=20$ and we obtain the unique solution:

$\boxed{v=20(1-e^{-0.1t})}$.

We can calculate the speed of the boat at any time *t*, e.g. at $t=5$ the speed is $v=20(1-e^{-0.5})=7.87$.

Another interesting observation we can make is that when $t\to\infty$ the term $e^{-0.1t}\to 0$ and thus the speed $v\to 20$. This is the maximum speed the boat can reach and is called **Terminal Velocity**. Note that the terminal velocity is *independent* of the constant *A* in the general solution.
This example also applies to a falling object in air such as a skydiver, etc.

POPULATION GROWTH

We define the size of a population (people living in a certain area, bacteria in a Petri dish, etc.) as a function of time as $x=x(t)$. The assumption is that the growth of the population is linearly dependent on the size of the population. This can be expressed as $\frac{dx}{dt}=kx$ where k is a constant.

This is a DE which we can solve with the 3 step approach:

Step 1 **Separate variables:**

$$\frac{1}{x}dx=kdt$$

Step 2 **Integrate both sides.**

$\int\frac{1}{x}dx=\int kdt$ or $\ln|x|=kt+c$. Now take the e-function on both sides as we did before: $e^{\ln|x|}=e^{kt+c}=e^{kt}\times e^{c}=Ae^{kt}$ (we converted the constant to simplify) or $\boxed{x=Ae^{kt}}$

This is the general DE with an unknown constant *A* describing the population size at any given time *t*.

(Note that we have two constants here: A is the integration constant and k the "model" constant relating growth to population size. If we want to find both we need two boundary conditions.).

Step 3 **Find the constants from a boundary condition.**

Suppose that we observe that the population doubles in 50 years (with bacteria this would go much faster). Translated into mathematics this means that if at $t=0$ the population size is x_0 then at $t=50$ $x=2x_0$. So we can write $x_0=Ae^0=A$ and $2x_0=Ae^{50k}=x_0e^{50k}$ or $e^{50k}=2$ hence $50k=\ln 2$ or $k=\frac{\ln 2}{50}=0.01386$. So the specific solution for the population size is $\boxed{x(t)=x_0e^{0.01386t}}$ where *t* is expressed in years. So we have found k and we know that $A=x_0$ is the population size at $t=0$.

We can now predict how the population will develop, e.g. when has the population trebled? Then $3x_0=x_0e^{0.01386t}$ hence $e^{0.01386t}=3$ and $t=\frac{\ln 3}{0.01386}=79.248$ year.

Unfortunately there is nothing like a "Terminal Population size" here. If *t* increases the growth is exponential and could best be described as a "runaway process".

A similar type of solution as above is applied in physics for e.g. radio-active decay (decreasing population size) and the charging or dis-charging of a capacitor. That is why these phenomena are often described with e-functions.

NEWTON'S LAW OF COOLING

States that the rate of cooling of a hot body is proportional to the temperature difference between its temperature and that of the surroundings. Hence $\frac{dT}{dt} = -k(T - T_0)$ where *T* is temperature and *t* is time. T_0 is the ambient temperature which usually is known. We have again a DE describing a physical situation.

Step 1 **Separate variables:**

$$\frac{dT}{T - T_0} = -kdt$$

Step 2 **Integrate both sides.**

$\int \frac{dT}{T - T_0} = -\int kdt$ or $\ln K|T - T_0| = -kt$ (we put the integration constant *K* in the logarithm) or $T - T_0 = Ae^{-kt}$ (we defined another constant $A = \frac{1}{K}$) or

$\boxed{T = T_0 + Ae^{-kt}}$ *A* is the general constant and *k* is the proportionality constant that we usually don't know either. So in this case we need two conditions to find a unique solution.

Step 3 **Find the constant from a boundary condition.**

If $T_0 = 20$ and we know the temperature at $t = 0$ e.g. $T = 70$ then

$70 = 20 + Ae^0 = 20 + A$ or $A = 50$.

As a second condition we might observe that at $t = 5$ $T = 40$. Then

$40 = 20 + 50e^{-5k}$ hence $e^{-5k} = \frac{40 - 20}{50} = 0.4$ and $k = \frac{\ln 0.4}{-5} = 0.183$.

The specific solution describing this situation is thus $\boxed{T = 20 + 50e^{-0.183t}}$. This allows us to calculate the temperature at any time *t*.

SOLVED PROBLEMS

1. The velocity v (ms^{-1}) of a skydiver increases according to the DE $\frac{dv}{dt} = 9.81 - 0.182v$ where 9.81 is the acceleration due to gravity and 0.182 gives the effect of air resistance (t is time in s).

    a. Solve the DE (i.e. find an expression for v as a function of time)
    b. Find the velocity of the skydiver 5 seconds after jumping from a stationary balloon.
    c. Determine the terminal velocity (limit of v when $t \to \infty$)

2. You invest $1000 with the RFC (Reliable Finance Company). It offers an interest rate of 16% per annum compounded continuously, i.e. If A is the total amount of money in dollars which accumulates after t years, then $\frac{dA}{dt} = 0.16A$

    a. Find an expression for A as a function of t.
    b. What is the total amount after two years
    c. In how many years will the amount total $10,000

3. The size y of a population of whales can be shown to satisfy the DE $\frac{dy}{dt} = 0.05y - 50000$ where t is time. The initial population is 1,100,000. Find the size of the population after 10 years.

4. Newton's law of cooling states that $\frac{dT}{dt} = -k(T - T_0)$ where T_0 is the ambient temperature and T is the temperature of a cooling object at time t. k is a constant. This formula is also used in forensic analysis to estimate the time of death of a person.

    The problem is as follows: Police arrive at a murder scene at 10:56 p.m. and measure the temperature of the victim's body to be 31^o C. One hour later the temperature has dropped to 30^o C. The room temperature where the body was found is 22^o C. The normal body temperature of a living person is 37^o C.
    Calculate the time of death.

5. A room of 200 m^3 contains air with a concentration of 0.2% CO_2 . Fresh air containing 0.05% CO_2 is pumped in at a rate of 50 m^3 per minute.
    Calculate the concentration of CO_2 after 5 minutes. (Hint: define x(t) is the amount of pure CO_2 in m^3 at time t).

Answers

1.

a. $\frac{dv}{dt}=9.81-0.182v$ or $\int\frac{dv}{9.81-0.182v}=\int dt$ or $\frac{\ln|9.81-0.182v|}{-0.182}=t+c$ or

$9.81-0.182v=Ae^{-0.182t}$ thus $v=\frac{9.81-Ae^{-0.182t}}{0.182}=53.9-Ke^{-0.182t}$.

At $t=0\to v=0$ hence $K=53.9$ and $\boxed{v=53.9(1-e^{-0.182t})}$

b. At $t=5$ $v=53.9(1-e^{-0.182\times 5})=32.2$ ms-1

c. For $t\to\infty$ $v\to 53.9$ ms-1

2.

a. $\frac{dA}{dt}=0.16A$ or $\int\frac{dA}{0.16A}=\int dt$ or $\frac{\ln|0.16A|}{0.16}=t+c$ or $\ln|0.16A|=0.16t+C$

or $0.16A=ke^{0.16t}$ or $A=Ke^{0.16t}$. At $t=0$ $A=1000$ thus $K=1000$ and $\boxed{A=1000e^{0.16t}}$

b. At $t=2$ $A=1000e^{0.16\times 2}=\$1377.13$

c. $10000=1000e^{0.16t}$ or $e^{0.16t}=10$ and thus $t=\frac{\ln 10}{0.16}=14.39$ years

3. $\frac{dy}{dt}=0.05y-50000$ or $\int\frac{dy}{0.05y-50000}=\int dt$ or $\frac{\ln|0.05y-50000|}{0.05}=t+c$ or

$0.05y-50000=Ce^{0.05t}$ or $y=\frac{50000+Ce^{0.05t}}{0.05}=10^6+We^{0.05t}$. At $t=0$ $y=1.1\times 10^6$

hence $W=10^5$ and $\boxed{y=10^5(10+e^{0.05t})}$

At $t=10$ $y=10^5(10+e^{0.5})=1{,}164{,}872$

4. First solve the DE: $\int\frac{dT}{T-T_0}=\int -kdt$ or $\ln K(T-T_0)=-kt$ or $K(T-T_0)=e^{-kt}$ or

$\boxed{T-T_0=Ae^{-kt}}$ where A is another constant.

We know that at $t=10:56=10\frac{56}{60}=10.9333$ $T=31$

And at $t=11:56=11\frac{56}{60}=11.9333$ $T=30$

Hence $31-22=9=Ae^{-k\times 10.9333}$ and $30-22=8=Ae^{-k\times 11.9333}$

Taking the logarithm on both sides and re-arranging gives:

$-k\times 10.9333=\ln 9-\ln A$ and similarly we have

$-k\times 11.9333=\ln 8-\ln A$

Subtracting gives $k=\ln 9-\ln 8=0.1178$ and substituting this in the first expression gives $\ln A=\ln 9+0.1178\times 10.9333$ or $A=32.6279$. At the time of death the body temperature was 37° C and thus $37-22=15=32.6279e^{-0.1178t}$ or

$e^{-0.1178t}=\frac{15}{32.6279}=0.4597$ and $t=\frac{\ln 0.4597}{-0.1178}=6.5969$ or $\boxed{6:36\text{ p.m.}}$

5. Let x denote the amount of CO_2 (in m^3) at any time t, the concentration then being $\frac{x}{200}\times 100\% = \frac{1}{2}x$ The amount of CO_2 pumped in per minute is $50\times\frac{0.05}{100}$ m^3 . The amount leaving per minute is $50\times\frac{\frac{1}{2}x}{100}$ m^3 .Hence the change in amount of CO_2 is $dx = 50\times\frac{0.05}{100} - 50\times\frac{\frac{1}{2}x}{100}$ or $\boxed{dx = (0.025 - \frac{x}{4})dt}$ Now solve this DE: $\int\frac{dx}{(0.025-\frac{x}{4})} = \int dt$ or $0.025 - \frac{x}{4} = Ce^{-\frac{t}{4}}$ or $x = 0.100 - Ke^{-\frac{t}{4}}$. At $t = 0$ the concentration is 0.2% hence $x(0) = 200\times\frac{0.2}{100} = 0.4$ so $0.4 = 0.1 - K$ or $K = -0.3$ thus the solution is $\boxed{x = 0.100 + 0.3e^{-\frac{t}{4}}}$. At $t = 5$ $x = 0.100 + 0.3e^{-\frac{5}{4}} = 0.186$ m^3 . And the concentration is then $\frac{0.186}{200}\times 100\% = 0.093\%$. And the "terminal" concentration? When $t\to\infty$ then $x\to 0.100$ and the concentration $\frac{0.100}{200}\times 100\% = 0.05\%$. This is what we would expect, isn't it?