Ngā Whetū Resources

Calculus Year 13 (NCEA Level 3)

Summary 5

Differentiation

Finding Limits

Calculator $\lim\limits_{x\to 0} x\cos x = 0$ Try values for x close to zero

Direct Substitution $\lim\limits_{x\to 8}(3x+2) = 26$

Algebraic Cancellation (eliminate common factors)

$$\lim_{x\to 5}\frac{x^2+x-30}{x^2-9x+20} = \lim_{x\to 5}\frac{(x-5)(x+6)}{(x-5)(x-4)} = \lim_{x\to 5}\frac{x+6}{x-4} = \frac{11}{1} = 11$$

Limits as $x\to\infty$ Divide each term by the highest power of x

$$\lim_{x\to\infty}\frac{x\ \ +3}{3x^2-1} = \lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{3x^2}{x^2}-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1+\frac{3}{x^2}}{3-\frac{1}{x^2}} = \frac{1}{3}$$

A limit does not exist at a point when the function value is different when approaching the point from below or from above. This includes vertical asymptotes.

Continuity Draw a graph of the function without lifting the pen
A function is dis-continuous at "holes", jumps and asymptotes.

Differentiability Graph of the function is smooth and continuous.
A function is not-differentiable at dis-continuities and "sharp corners".

Differentiation from first principles

Solve $f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Example: $f(x) = x^2+2$ Then $f'(x) = \lim\limits_{h\to 0}\frac{(x+h)^2+2-(x^2+2)}{h} =$

$$\lim_{h\to 0}\frac{x^2+2hx+h^2+2-x^2-2}{h} = \lim_{h\to 0} 2x+h = 2x$$

Differentiation of Polynomials

$y = ax^n$ then $y' = a.n.x^{n-1}$

Chain Rule for composite functions $y = f(g(x))$ then $y' = f'(g(x)).g'(x)$

Example: $y = (5x-7)^3$ then $y' = 3(5x-7)^2.5 = 15(5x-7)^2$

Product Rule $y = f.g$ then $y' = f'.g + f.g'$

Example: $y = 2x^2(3x+1)$ then $y' = 4x(3x+1) + 2x^2.3 = 12x^2 + 4x + 6x^2 = 18x^2 + 4x$

Quotient Rule $y = \frac{f}{g}$ then $y' = \frac{f'.g - f.g'}{g^2}$

Example: $y = \frac{8x-1}{x^3+1}$ then

$$y' = \frac{8(x^3+1) - (8x-1).3x^2}{(x^3+1)^2} = \frac{8x^3 + 8 - 24x^3 + 3x^2}{(x^3+1)^2} = \frac{-16x^3 + 3x^2 + 8}{(x^3+1)^2}$$

Differentiation of Functions

Exponential Function $y = e^x$ (>0 for all *x*) then $y' = e^x$

Definition of Logarithm $^{10}\log 100 = 2$ because $10^2 = 100$ (10 is the base)

We will use **ln** which has the number *e* as base $^{e}\ln(e^x) \equiv x$.

In words: the logarithm of a number is the power to which you must raise the base to obtain the number.

Properties of Logarithm

$\ln(a.b) = \ln a + \ln b$ (change product into sum)

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ (change quotient into difference)

$\ln(a^n) = n.\ln a$ (change power into product)

Differentiation of Logarithm

$y = \ln x$ then $y' = \frac{1}{x}$

Trig Functions

$y = \sin x$	$y' = \cos x$
$y = \cos x$	$y' = -\sin x$
$y = \tan x$	$y' = \frac{1}{\cos^2 x} = \sec^2 x$

Combine all these with Chain Rule, Product Rule and/or Quotient Rule where necessary.

Geometric Properties of Differentiation

$y = f(x)$ First Derivative $y' = f'(x)$ defines value of gradient of $f(x)$ at each point.

Example: $y = x^2 - 5x + 4$ then $y' = 2x - 5$

At the point (3,-2) the gradient is $2.3 - 5 = 1$

Equation of the Tangent

$y - y_1 = m(x - x_1)$ hence $y - (-2) = 1.(x - 3)$ or $y = x - 5$

Equation of the Normal

Same procedure but gradient is inverse reciprocal: $m_{normal} = -\dfrac{1}{m_{tangent}}$

Implicit Differentiation

y is an implicit function of x when it cannot be expressed explicitly in the form $y = f(x)$

Example: $2x^2y + 3xy^2 = 16$

Differentiate this as (using Chain and Product Rule): $4xy + 2x^2\dfrac{dy}{dx} + 3y^2 + 3x.2y\dfrac{dy}{dx} = 0$

Now make $\dfrac{dy}{dx}$ subject: $(2x^2 + 6xy)\dfrac{dy}{dx} = -4xy - 3y^2$ Hence $\dfrac{dy}{dx} = \dfrac{-4xy - 3y^2}{2x^2 + 6xy}$

Stationary Points

See separate hand-out ***Stationary Points***